Quadratic Constraints for Local Stability Analysis of Quadratic Systems

2022 IEEE Conference on Decision and Control, Cancun, Mexico

CDC22
Conference on Decision and Control
Dec. 6-9, 2022 | Cancún, Mexico

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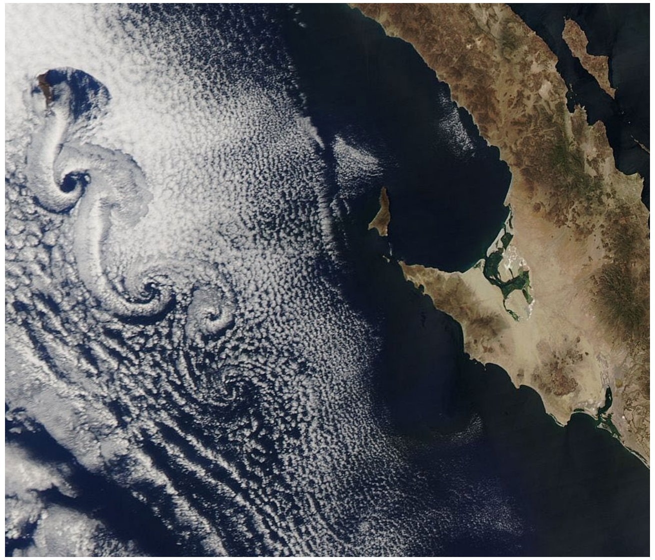
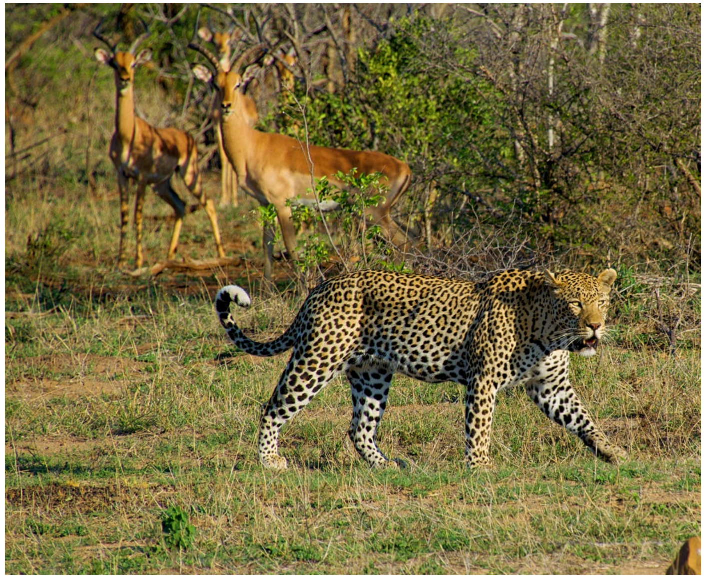
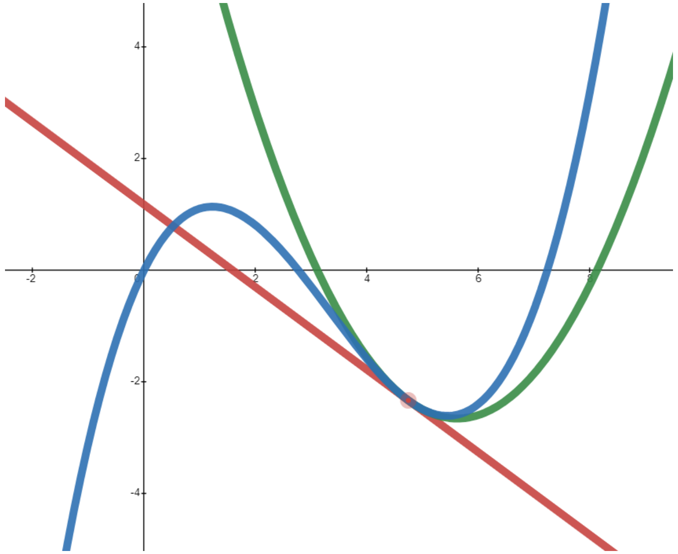
Acknowledgement: *This work was funded by the US Army Research Office*

Motivation: Quadratic Systems

- Navier-stokes equations
- Population dynamics
- Taylor approximation







$$\dot{x}(t) = Ax(t) + \phi(x(t)), \text{ where } \phi(\cdot) \text{ is a quadratic function}$$

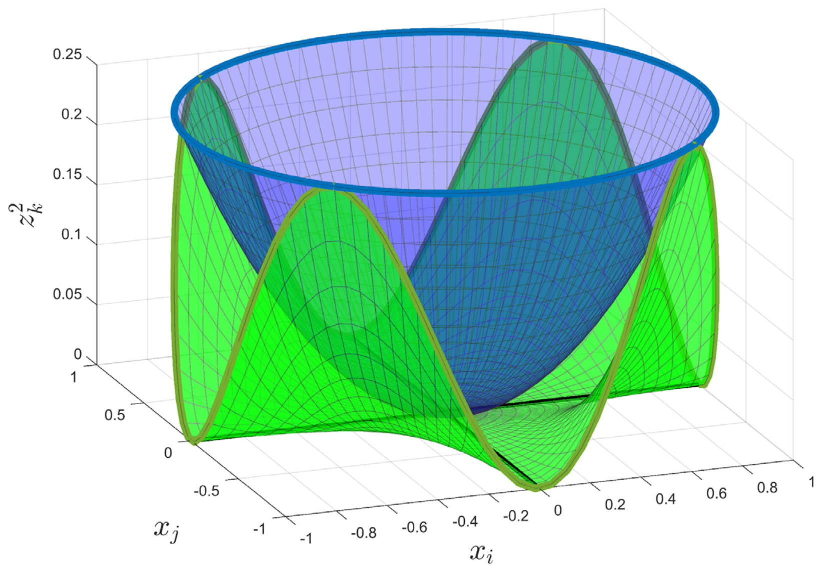
How to analyze the local stability of such nonlinear quadratic systems?

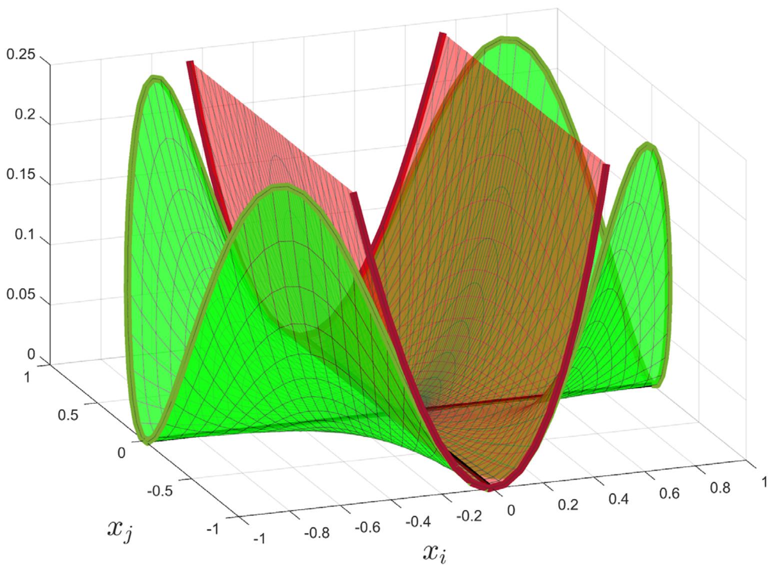
Key Takeaways

1. Proposed **new quadratic constraint (QC)** to characterize quadratic nonlinearities

2. **Reduced conservatism** in stability analysis using newly proposed QCs

3. The new QCs can be applied to other **analysis involving dissipation inequality** such as prformance, reachability, robustness





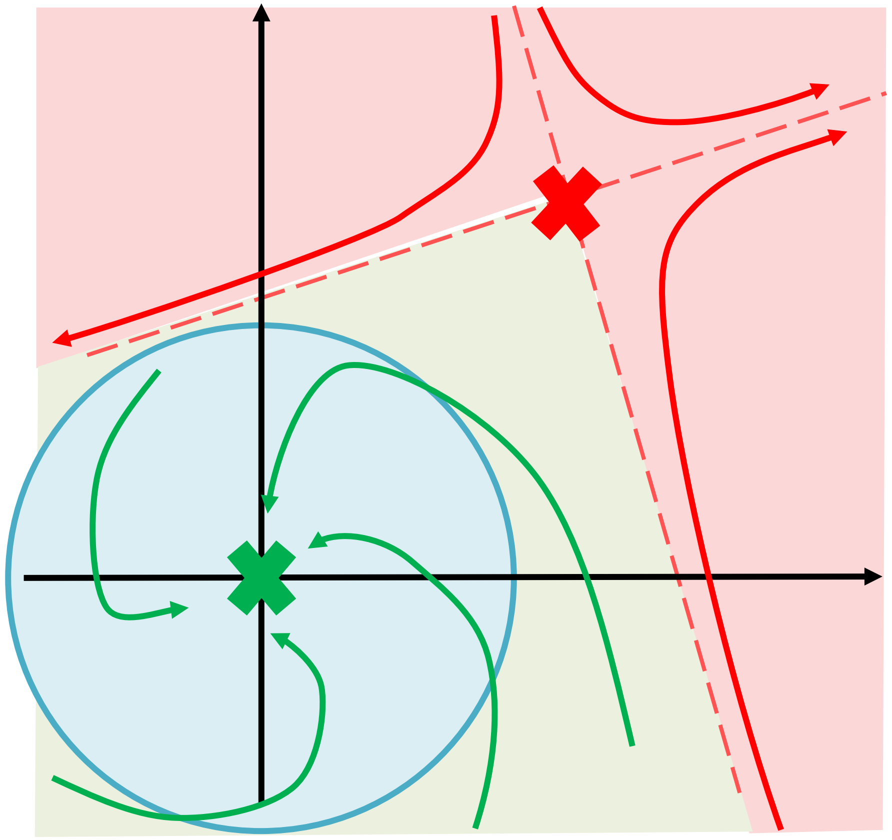
Outline

- Motivation
- Problem Formulation
- Region of Attraction Estimation using Quadratic Constraints
- Quadratic Constraints for Quadratic Functions
- Numerical Example
- Conclusion

Problem Formulation

- Given a quadratic nonlinear ODE
  - $A \in \mathbb{R}^{n \times n}$ is Hurwitz.
  - ϕ is homogenous quadratic polynomial
  - Multiple equilibrium points.

$$\dot{x}(t) = Ax(t) + \phi(x(t))$$

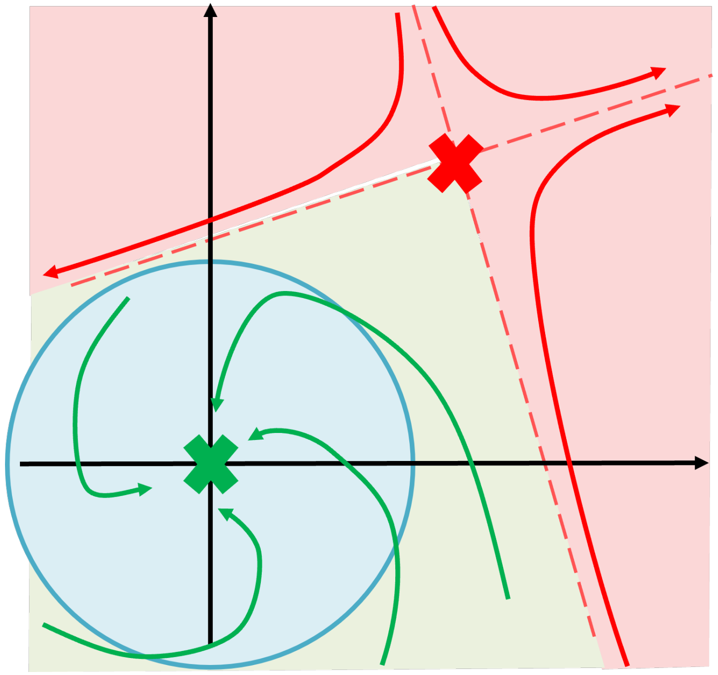


- Region of attraction (ROA) of the origin*:
  - Initial conditions that converge to the origin
- Goal: find a largest spherical ROA estimate
  - Radius as the metric of the size of ROA estimate

Other equilibrium can be shifted to origin and form another quadratic system by change of coordinates.

Methods for ROA Estimation

- Direct simulation
  - Easy to verify stability of initial conditions
  - No guaranteed over a region
- Lyapunov stability using sum-of-square optimization [1][2]
  - Good estimation with high-degree Lyapunov function
  - Suffer from curse of dimensionality
- **Absolute stability using quadratic constraints (QCs) [3][4][5]**
  - Moderate scalability
  - **Conservative estimation**

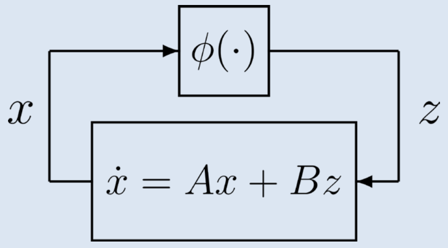


[1] P. Parrilo, “Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization,” Thesis, 2000
[2] U. Topcu, A. Packard, P. Seiler, and G. Balas, “Help on SOS,” IEEE CSM, 2010
[3] C. Liu and D. F. Gayme, “I/O inspired method for permissible perturbation amplitude of transitional wall-bounded shear flows,” Phys. Rev. E, 2020
[4] A. Kalur, T. Mushtaq, P. Seiler, and M. S. Hemati, “Estimating ROA for Transitional Flows using QC,” IEEE L-CSS, 2021
[5] L. F. Toso, R. Drummond, and S. R. Duncan, “Regional stability analysis of transitional fluid flows,” IEEE L-CSS, 2022

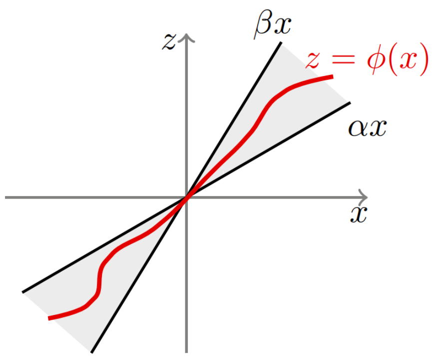
ROA Estimation using Quadratic Constraints

Separating Nonlinearity → Characterizing Nonlinearity → Formulating Local Stability Condition → Finding Largest Spherical ROA

Lur’e System



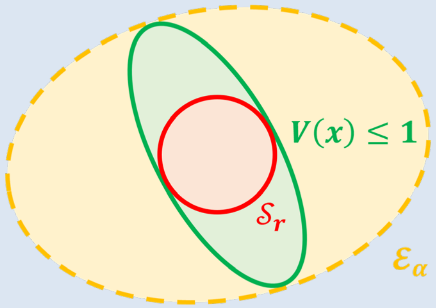
Quadratic Constraints



S-Procedure

$$\dot{V}(x) < 0$$
$$\forall x \in \mathcal{E}_\alpha$$

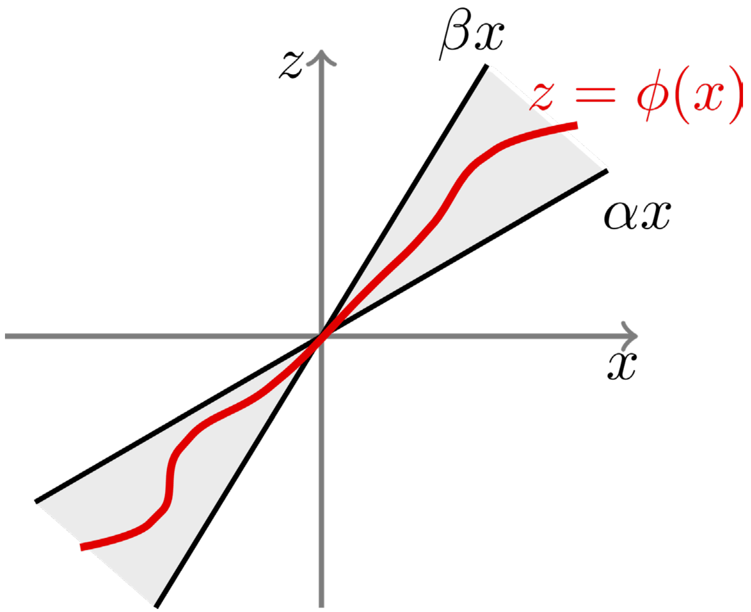
Semidefinite Program



[6] V. M. Popov, “Absolute stability of nonlinear systems of automatic control,” Automation Remote Control, 1962
[7] V. A. Yakubovich, “Frequency condition of absolute stability …,” Avtomat. i Telemekhan., 1967
[8] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE TAC*, 1997
[9] J. Veenman, C. W. Scherer, and H. Köroğlu, “Robust stability and performance analysis based on IQC.” *European Journal of Control*, *2016*

Quadratic Constraints

- Consider nonlinearity $\phi: \mathbb{R} \to \mathbb{R}$ is (locally) sector bounded in $[\alpha, \beta]$:
  - $z = \phi(x)$
  - $\boldsymbol{(z - \alpha x)(\beta x - z) \geq 0 \;\; \forall x \in \mathcal{E}}$



- Characterize ϕ by its input and output signals:

$$\begin{bmatrix} x \\ z \end{bmatrix}^\top M_{\alpha,\beta} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0, \; z = \phi(x), \forall x \in \mathcal{E}$$

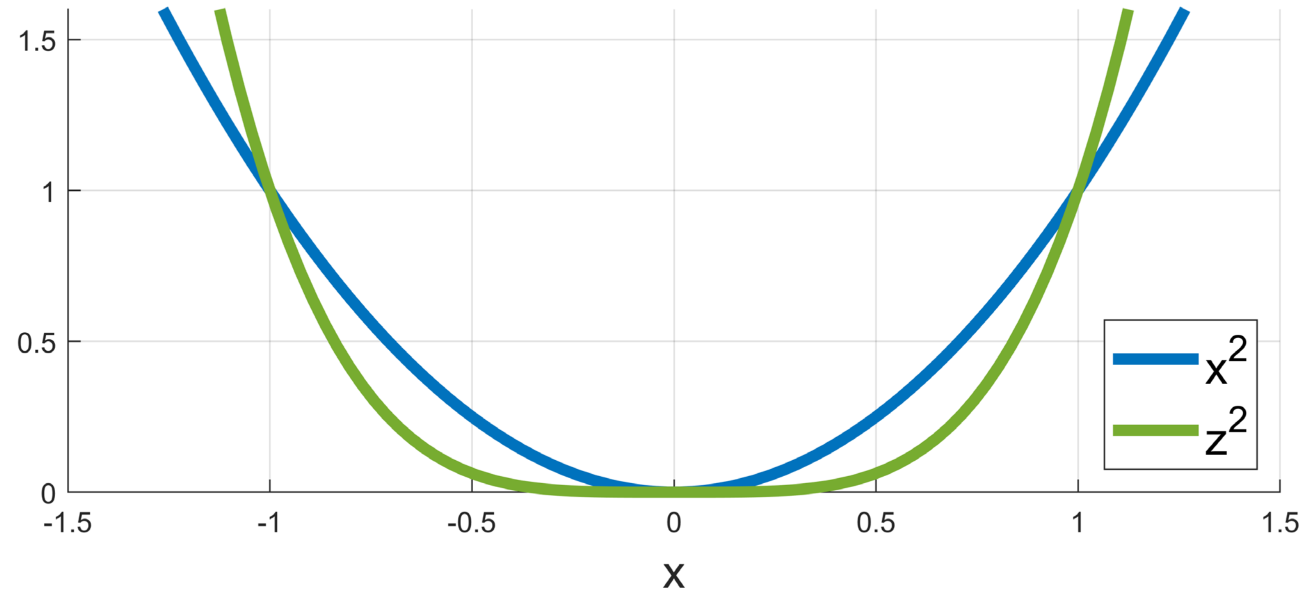
- **A tighter QC provides a more accurate description of $\boldsymbol{\phi}(\cdot)$ and gives a less conservative analysis.**

Existing QCs for Quadratic function

- [3] proposed QCs to bound $\phi(\cdot)$ in a spherical local region $\{x|x^\top x \leq \alpha^2\}$.
- [4] applied Cauchy-Schwarz inequality to generalized QCs to an ellipsoidal local region $\{x|x^\top Px \leq \alpha^2\}$, named **CSQC**.

- For example, $z = \phi(x) = x^2$
  - CSQC in $\{x^\top x \leq 1\}$

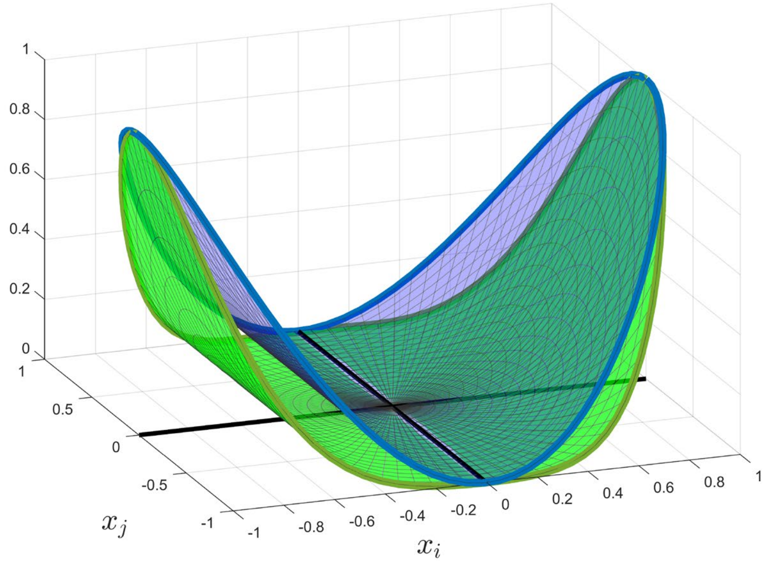
$$\begin{bmatrix} x \\ z \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = x^2 - z^2 \geq 0$$



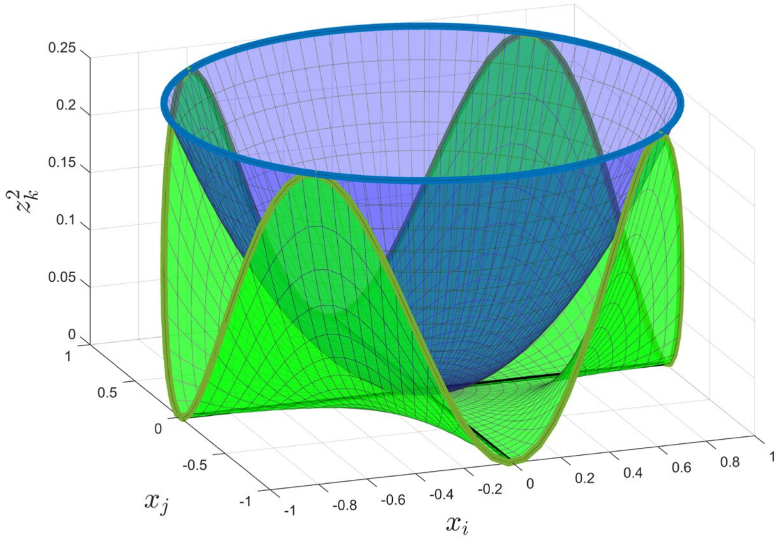
[3] C. Liu and D. F. Gayme, "I/O inspired method for permissible perturbation amplitude of transitional wall-bounded shear flows," Phys. Rev. E, 2020
[4] A. Kalur, T. Mushtaq, P. Seiler, and M. S. Hemati, "Estimating ROA for Transitional Flows using QC," IEEE L-CSS, 2021

Conservatism of CSQC [4]

- $z_1 = x_i^2$
  - Positive definite
- CSQC in $\{x^\top x \le 1\}$
  - $x_i^2 - z_1^2 \ge 0$



- $z_2 = x_i x_j,\ i \ne j$
  - Sign-indefinite
- CSQC in $\{x^\top x \le 1\}$
  - $\frac{1}{4}\left(x_i^2 + x_j^2\right) - z_2^2 \ge 0$



CSQC is loose along $x_i = 0$ and $x_j = 0$

[4] A. Kalur, T. Mushtaq, P. Seiler, and M. S. Hemati, "Estimating ROA for Transitional Flows using QC," IEEE L-CSS, 2021

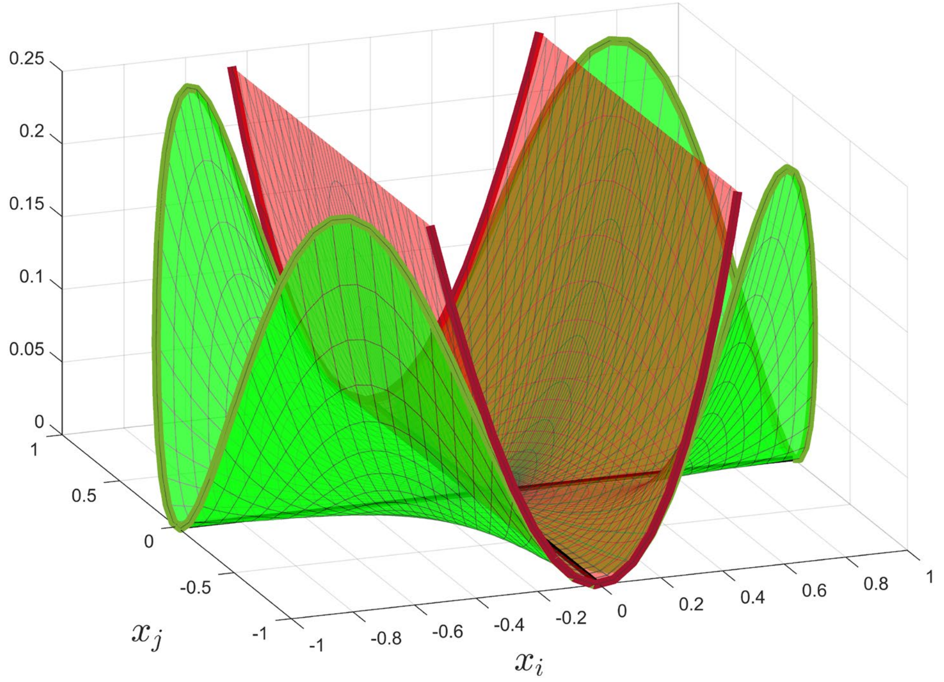
New QCs on $z_2 = x_i x_j$

- CSQC is loose along $x_i = 0$

- **Valley QC** along $x_i = 0$:

$$x_i^2 - z_2^2 = x_i^2(1 - x_j^2) \geq 0 \quad \forall x \in \{x^\top x \leq 1\}$$

  - Tight along $x_i = 0$
  - Loose elsewhere

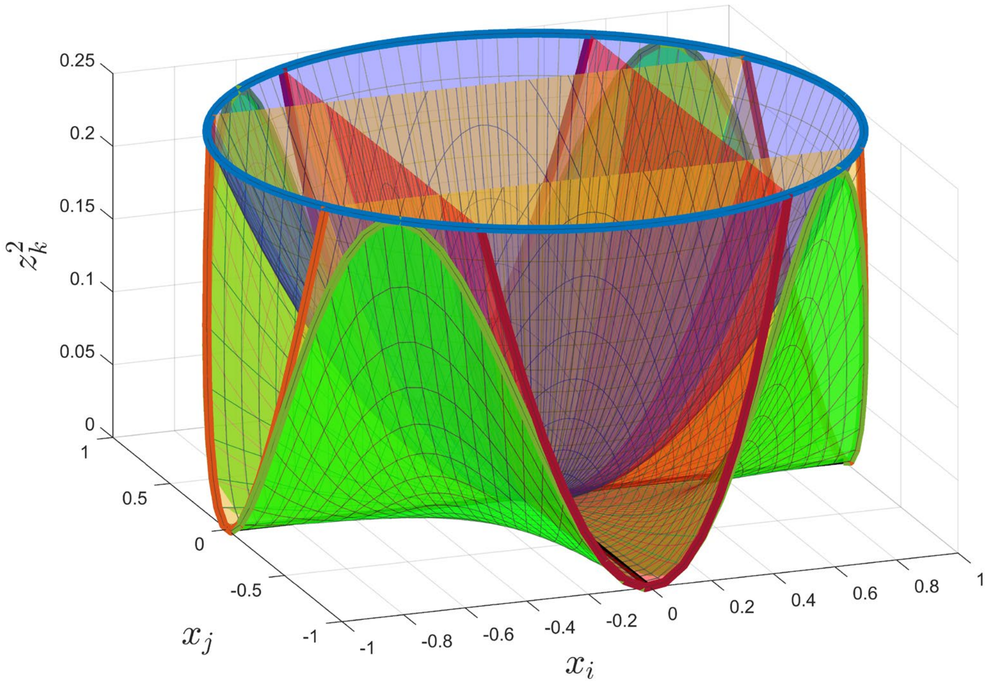


- There exists another Valley QC along $x_j = 0$

CSQC and Valley QCs jointly Bound $z_2 = x_i x_j$

- Monomial $z_2 = x_i x_j$

- Less conservative characterization by CSQC, Valley QC 1, and Valley QC 2



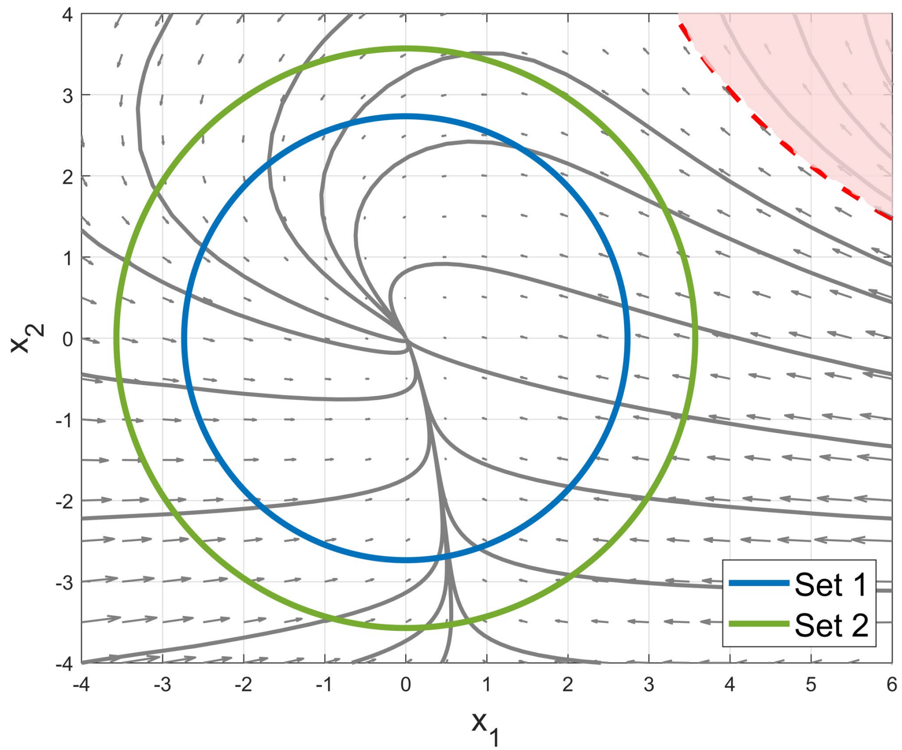
- Generalization of Valley QC:
  - Ellipsoidal region $\{x^\top E x \leq \alpha^2\}$
  - Quadratic fun. with Hessian being **rank-2 and sign-indefinite**
  - Quadratic fun. with Hessian being **rank-3 and sign-indefinite**
  - Cross-product of quadratic functions (inspired by [5])

See paper for more details

[5] L. F. Toso, R. Drummond, and S. R. Duncan, "Regional stability analysis of transitional fluid flows," IEEE L-CSS, 2022

Numerical Examples – 2-State Example [6]

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -50 & -16 \\ 13 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 13.8 \\ 5.5 \end{bmatrix} x_1 x_2.$$



- Largest spherical ROA ≈ 4.95
- ROA estimation using two sets of QC
  - Set 1: $r_1^* = 2.7355$
    - 1* CSQC
  - Set 2: $r_2^* = 3.5224$
    - 1* CSQC, 2* Valley QCs
- The use of Valley QCs produce a better estimation.
- A 3-state example is presented in the paper.

[6] F. Amato, C. Cosentino, and A. Merola, “On the region of asymptotic stability of nonlinear quadratic systems,” Mediterranean CCA, 2006

Conclusions

1. Proposed **new quadratic constraints** to characterize quadratic nonlinearities
2. **Reduced conservatism** in stability analysis using newly proposed QCs
3. The new QCs can be applied to other **analysis involving dissipation inequality**

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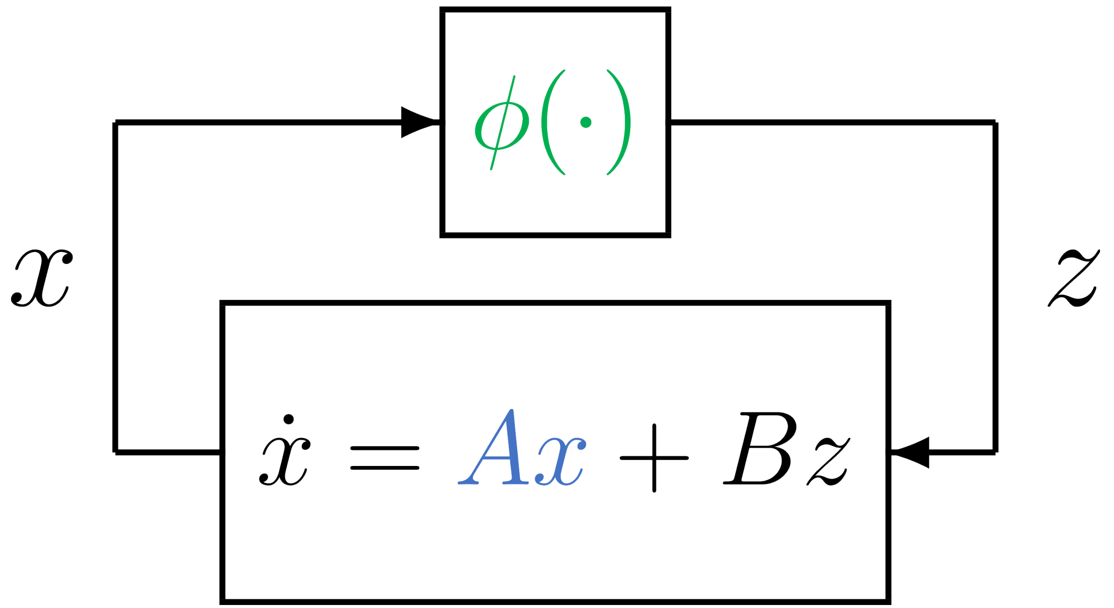
BACKUP SLIDES

Lur'e Decomposition

- System: $\dot{x} = Ax + \phi(x)$
- Separate the linear part and nonlinear part of a system into feedback interconnection:

$$\dot{x} = Ax + Bz$$
$$Bz = \phi(x)$$

  - B is a matrix
  - z is a vector



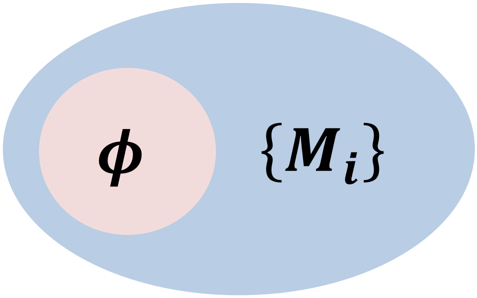
- $z = \phi(x)$ and $B = I_{n\times n}$ in [3].
- In our work:
  - z as quadratic monomials
  - B is the corresponding matrix

Quadratic Constraints (QCs)

- In general, a nonlinearity $z = \phi(x)$ is hard to analyze

- Use QCs to describe the input-output behavior of ϕ

$$\begin{bmatrix} x \\ z \end{bmatrix}^\top M_i \begin{bmatrix} x \\ z \end{bmatrix} \geq 0\,, \forall z = \phi(x), x \in \mathcal{E}_\alpha$$



- If the Lur’e system is quadratically stabile with the set of QCs, the system with the actual nonlinearity ϕ is stable

- A set of QCs that describe ϕ more accurately is desired
  - Analyzing with a smaller set of possible nonlinearities

QC on a Quadratic Nonlinearity

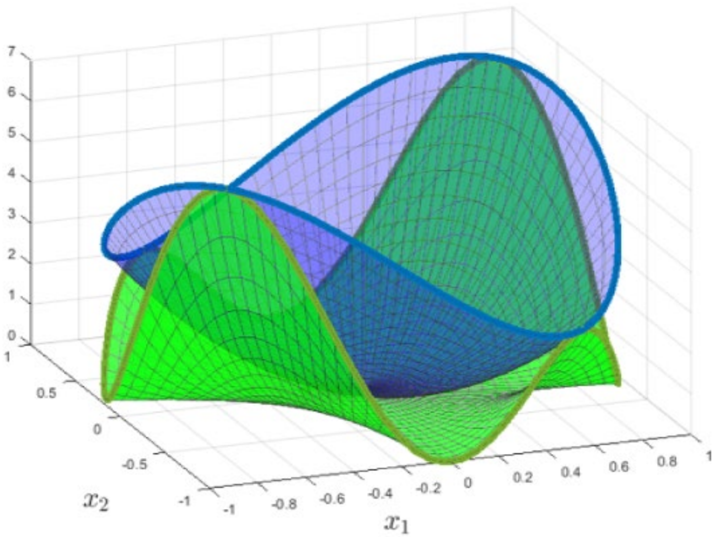
- A quadratic nonlinearity $z_i = \phi_i(x) = x^\top Q_i x$
  - E.g., $z_i = x_1^2 + 4x_1x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

In the unit circle $\{\boldsymbol{x}^\top \boldsymbol{x} \leq \mathbf{1}\}$

- Cauchy-Schwartz QC (CSQC) [3]
  - In a local region $\mathcal{E}_\alpha = \{x : x^\top E x \leq \alpha^2\}$

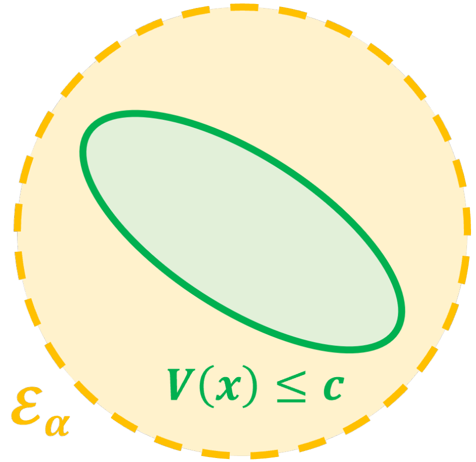
$$\begin{bmatrix} x \\ z \end{bmatrix}^\top \underbrace{\begin{bmatrix} \alpha^2(Q_i E^{-1} Q_i) & 0 \\ 0 & -e_i e_i^\top \end{bmatrix}}_{M(E,\alpha)} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0 \,, \; \forall x \in \mathcal{E}_\alpha$$

$$\Rightarrow \; x^\top\big(\alpha^2(Q_i E^{-1} Q_i)\big)x - z_i^2 \geq 0$$



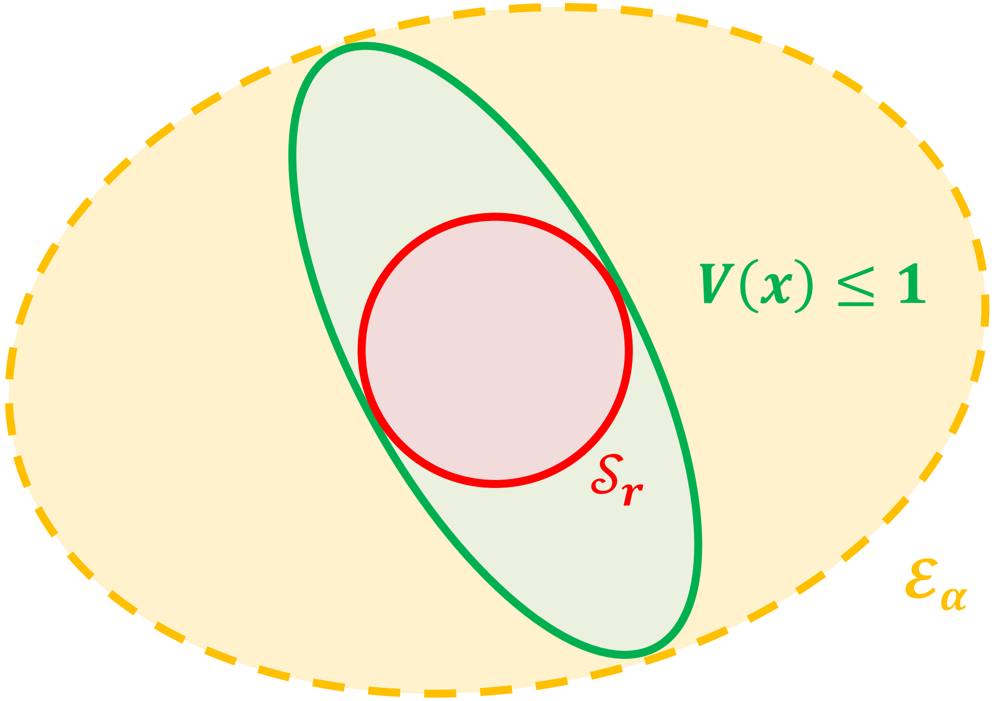
Quadratic Stability Condition

- Consider the Lur'e system $\begin{cases} \dot{x} = Ax + Bz \\ z = \phi(x) \end{cases}$, where the nonlinearity $z = \phi(x)$ satisfies the local QC $\{M_i\}$ in $\mathcal{E}_\alpha$.
- If $\exists P \succ 0 \; and \; \xi_i \geq 0 \; such \; that \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} + \sum_i \xi_i M_i \prec 0$
  - $V(x) = x^\top P x$ is a quadratic Lyapunov function
  - $\dot{V}(x) < 0 \; \forall x \in \mathcal{E}_\alpha$ (can be shown by S-procedure [7])
  - $\{x : V(x) \leq c\} \subseteq \mathcal{E}_\alpha$ is an ROA



Finding the Largest ROA Estimate

Maximize r
ξ, P, r, E, α
Subject to:
$\mathcal{S}_r \subseteq \{V(x) \leq 1\} \subseteq \mathcal{E}_\alpha$
$\dot{V}(x) < 0, \ \forall x \in \mathcal{E}_\alpha$



- By Lyapunov stability
  - $\{V(x) \leq 1\}$ is an invariant set
  - **The sphere $\mathcal{S}_r$ is an ROA estimate**

- Can be solved by SDPs with iteratively update $\{E, \alpha\}$ [3]