Robust Control Barrier Functions with Sector-Bounded Uncertainties

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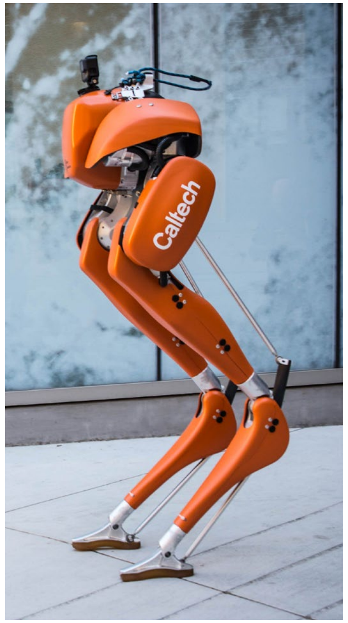
Motivation: Safety Critical Systems

Robotics

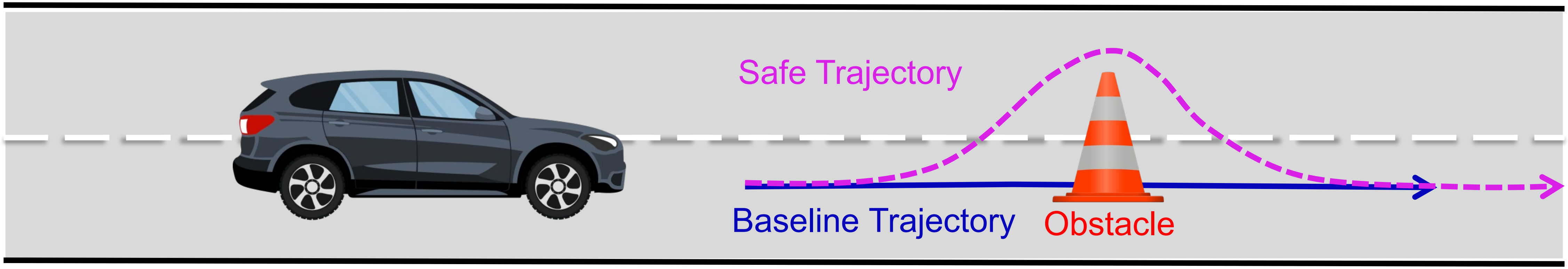


Nuclear Power Plants

Unmanned Aerial Vehicles

Autonomous Driving

Example: Obstacle Avoidance



Control Barrier Functions (CBF) [1,2] are used as an approach to guarantee safety. However, CBF requires a perfect model.

In practice, reduced order/linearized vehicle models are used for control design.

Need for safety-critical control methods that account for **model mismatch/uncertainties**.

[1] Ames, Coogan, Egerstedt, Notomista, Sreenath, Tabuada. Control barrier functions: theory and applications. IEEE ECC, 2019.
[2] Ames, Xu, Grizzle, Tabuada. Control barrier function based quadratic programs for safety critical systems. IEEE TAC, 2016.

Key Takeaways

1. A new Robust Control Barrier Function (RCBF) approach to handle **sector-bounded nonlinearities at the plant input**.

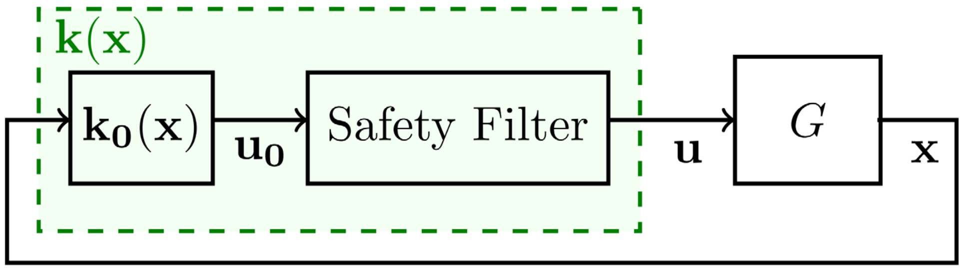
2. Propose an optimization problem to guarantee robust-safety. Recast this problem into **Second-Order Cone Program (SOCP)**.

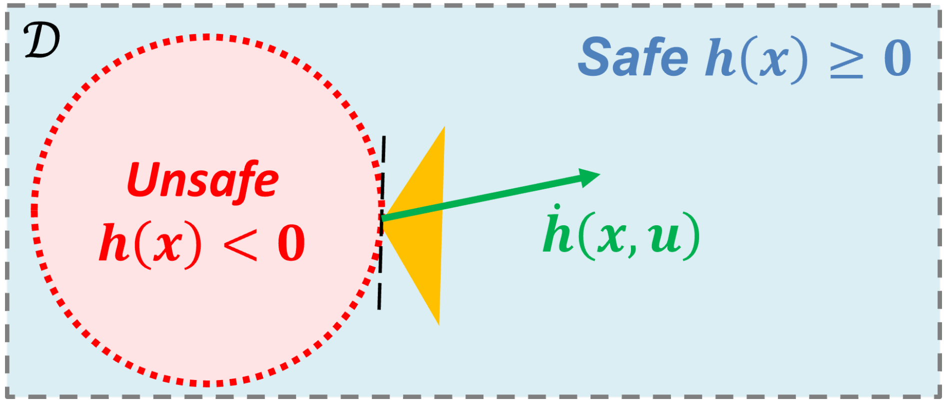
3. Conjecture: The solution of the SOCP are a **locally Lipschitz continuous function of the state**. Proof is given for the scalar input case.

Outline

- Background
- Problem Formulation
- Main Results
- Numerical Example: Lateral Vehicle Control
- Conclusion

Background: Control Barrier Functions (CBFs)





Safety Filter designed using CBF-QP [1]:

$$\mathbf{u}^*(\mathbf{x}) = \arg\min_{\mathbf{u}\in\mathcal{U}} \frac{1}{2}\|\mathbf{u}-\mathbf{u_0}\|_2^2$$
$$\text{s.t. } L_{\mathbf{f}}h(\mathbf{x}) + L_{\mathbf{g}}h(\mathbf{x})\,\mathbf{u} \geq -\eta(h(\mathbf{x}))$$

- G is a control-affine system:
  $$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)\mathbf{u}(t)$$
- k_0 is baseline controller

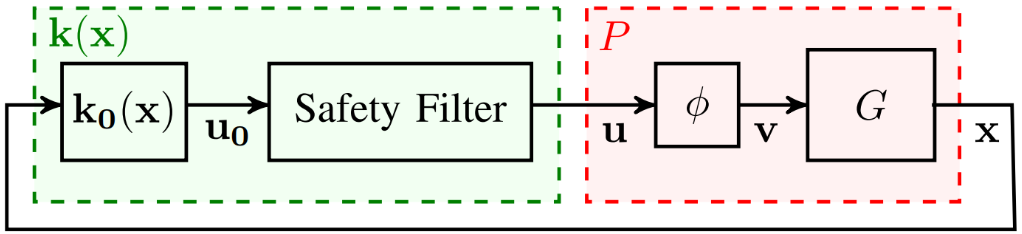
- Safe-set is defined as:
  $$\mathcal{C} \coloneqq \{x \in \mathcal{D} \subset \mathbb{R}^n : h(x) \geq 0\}$$
- h is a CBF [1] if there exists $\eta \in \mathcal{K}_{\infty,e}$ s.t.
  $$\sup_{\mathbf{u}\in\mathcal{U}} \left[L_{\mathbf{f}}h(\mathbf{x}) + L_{\mathbf{g}}h(\mathbf{x})\,\mathbf{u}\right] \geq -\eta(h(\mathbf{x}))$$

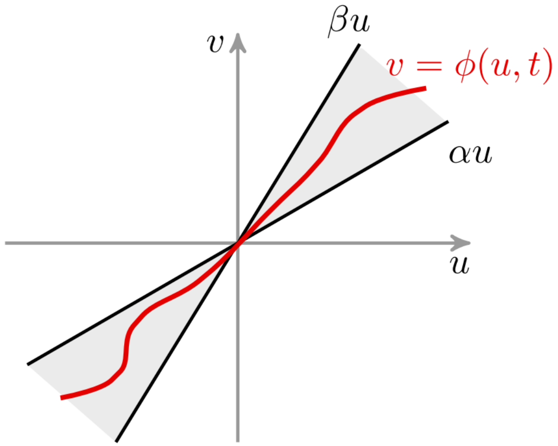
- Minimal perturbation to nominal control
- Enforces safety as a hard requirement

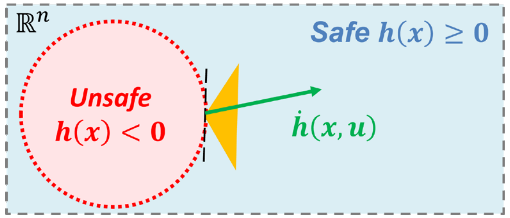
Limitation
- Require the exact model of the system

[1] Ames, Coogan, Egerstedt, Notomista, Sreenath, Tabuada. Control barrier functions: theory and applications. IEEE ECC, 2019.

Problem Formulation







Uncertain Plant P:

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)v(t),\ x(0) = x_0$$
$$v(t) = \phi(u(t), t)$$

Sector-Bounded Nonlinearity ϕ:

$$[v(t) - \alpha u(t)]^\top [\beta u(t) - v(t)] \geq 0, \forall t \geq 0$$

Time-varying, memoryless uncertainties

Safety:

$$\mathcal{C} \coloneqq \{x \in \mathcal{D} \subset \mathbb{R}^n : h(x) \geq 0\}$$

Goal: Design a Safety Filter such that if $\mathrm{x}_0 \in \mathcal{C}$ then the closed-loop system remains safe.

Uncertainty Mapping (Loop Shifting)

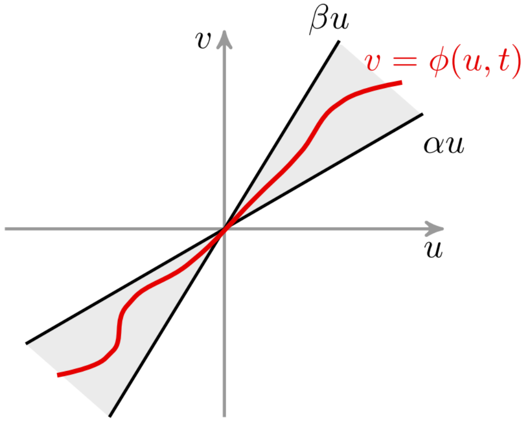
Sector-bound nonlinearity ϕ:

$$[\mathbf{v}(t) - \alpha\mathbf{u}(t)]^\top [\beta\mathbf{u}(t) - \mathbf{v}(t)] \geq 0, \;\; \forall t \geq 0.$$

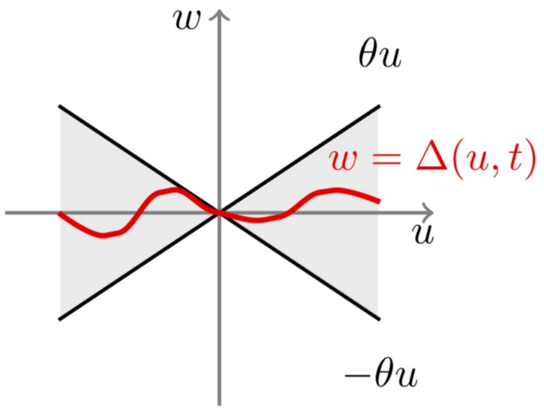
Norm-bound nonlinearity Δ:

$$\|\mathbf{w}(t)\|_2 \leq \theta \|\mathbf{u}(t)\|_2, \;\; \forall t \geq 0 \text{ where } \theta := \frac{\beta - \alpha}{\beta + \alpha}$$

$$\mathbf{w}(t) \in \mathcal{W}(\mathbf{u}(t))$$



$$v = \frac{\beta + \alpha}{2}(u + w)$$



The uncertain system P is:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{x}(t))\,\mathbf{v}(t)$$

$$\mathbf{v}(t) = \phi(\mathbf{u}(t), t)$$

The mapped uncertain system is given by:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \tilde{\mathbf{g}}(\mathbf{x}(t))\,(\mathbf{u}(t) + \mathbf{w}(t))$$

$$\mathbf{w}(t) = \Delta(\mathbf{u}(t), t)$$

where $\tilde{\mathbf{g}}(\mathbf{x}) := \frac{1}{2}(\alpha + \beta)\,\mathbf{g}(\mathbf{x})$

[1] Loop Shifting, Section 6.5 in Khalil, Nonlinear Systems, Prentice Hall, 2002.

Robust Control Barrier Functions (RCBFs)

h is a Robust CBF if there exists $\eta \in \mathcal{K}_{\infty,e}$ s.t.

$$\sup_{\mathbf{u}\in\mathcal{U}} \inf_{\mathbf{w}\in\mathcal{W}} [L_{\mathbf{f}}h(\mathbf{x}) + L_{\tilde{\mathbf{g}}}h(\mathbf{x})(\mathbf{u}+\mathbf{w})] \geq -\eta(h(\mathbf{x})) \qquad \text{where } \mathcal{W} := \{\|\mathbf{w}(t)\|_2 \leq \theta\|\mathbf{u}(t)\|_2\},\ \forall t \geq 0.$$

Worst-case nonlinear input: $\mathbf{w}^*(\mathbf{u}) = -\theta\|\mathbf{u}\|_2 \dfrac{L_{\tilde{\mathbf{g}}}h(\mathbf{x})^\top}{\|L_{\tilde{\mathbf{g}}}h(\mathbf{x})\|_2}$

RCBF can be rewritten as $\sup_{\mathbf{u}\in\mathcal{U}} [L_{\mathbf{f}}h(\mathbf{x}) + L_{\tilde{\mathbf{g}}}h(\mathbf{x})(\mathbf{u}+\mathbf{w}^*(u))] \geq -\eta(h(\mathbf{x}))$

Safety Filter design using RCBF:

$$\begin{aligned}\mathbf{u}^*(\mathbf{x}) = \arg\min_{\mathbf{u}\in\mathcal{U}} & \frac{1}{2}\|\mathbf{u}-\mathbf{u_0}\|_2^2 \\ \text{s.t. } & L_{\mathbf{f}}h(\mathbf{x}) + \eta(h(\mathbf{x})) + L_{\tilde{\mathbf{g}}}h(\mathbf{x})(\mathbf{u}+\mathbf{w}^*(\mathbf{u})) \geq 0\end{aligned}$$

Not an QP!
w^* depends on $\|u\|_2$

Online Implementation

Define a slack variable $q := \frac{1}{2}\mathbf{u}^\top \mathbf{u}$ and rewrite the optimization problem as:

$$\begin{bmatrix} \mathbf{u}^*(\mathbf{x}) \\ q^*(\mathbf{x}) \end{bmatrix} = \arg\min_{\mathbf{u}\in\mathcal{U},q} \left[q - \mathbf{u_0}^\top \mathbf{u}\right]$$

$$\text{s.t. } \theta \|L_{\tilde{\mathbf{g}}} h(\mathbf{x})\|_2 \|\mathbf{u}\|_2 \leq L_{\mathbf{f}} h(\mathbf{x}) + \eta(h(\mathbf{x})) + L_{\tilde{\mathbf{g}}} h(\mathbf{x})\mathbf{u}$$

$$\left\| \begin{bmatrix} \sqrt{2}\,\mathbf{u} \\ q-1 \end{bmatrix} \right\|_2 \leq q+1$$

Second-Order Cone Program (SOCP)

Main Results:

1) Existence of a Robust CBF ensures Robust Control Invariance. Proof follows from generalizations of Nagumo's theorem [1].
2) For scalar input case, the optimization solution is a locally Lipschitz Continuous function of the state [2].

[1] Chapter 4 of Blanchini, Miani. Set-theoretic methods in control, 2008.
[2] Weaver. Lipschitz algebras. World Scientific, 2018.

Extensions

- Robust Exponential CBFs (RECBF)

- Unifying with Robust Control Lyapunov Functions (CLFs) [1]

- Parametric Uncertainty

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \left[\mathbf{g}_0(\mathbf{x}) + \sum_{i=1}^{n_p} \mathbf{g}_i(\mathbf{x})\delta_i\right]\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

$$|\delta_i| \leq \theta_i$$

$$\mathbf{w}_i(t) := \delta_i \mathbf{u}(t) \qquad \|\mathbf{w}_i(t)\| \leq \theta_i \|\mathbf{u}(t)\|_2$$

Special case:

If $n_p = 1$ and $\mathbf{g_1}(\mathbf{x}) = \mathbf{g_0}(\mathbf{x})$, then
$\dot{\mathbf{x}} = \mathbf{f_0}(\mathbf{x}) + \mathbf{g_0}(\mathbf{x})\left(\mathbf{u} + \mathbf{w}\right)$
$\mathbf{w} = \delta_1 \mathbf{u}$ and $|\delta_1| \leq \theta_1$

Gain variation at the plant input as in gain-margin calcuation.

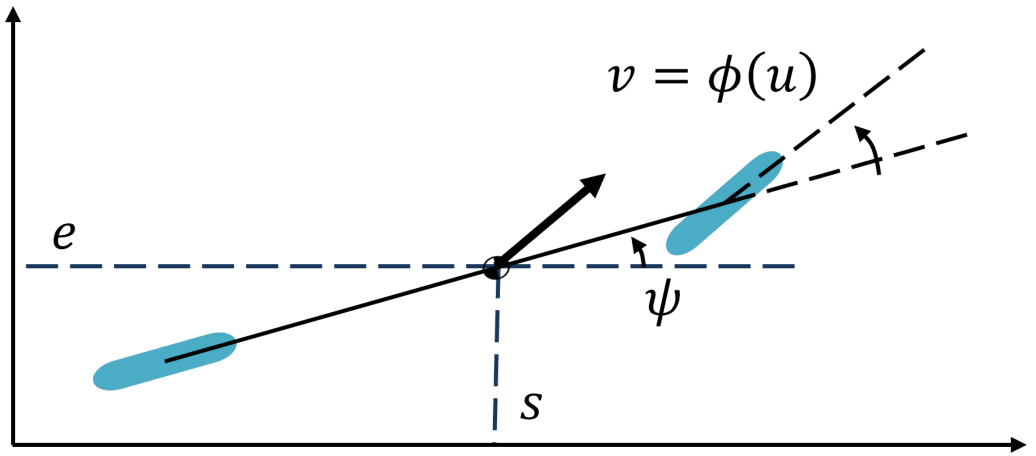
[1] Freeman, Kokotovic, Robust Nonlinear Control Design, 1996.

Numerical Example: Lateral Vehicle Control

Lateral vehicle dynamics are linearized at a constant longitudinal speed [1]

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}v(t), \ \ \mathbf{x}(0) = \mathbf{x_0}$$
$$v(t) = \phi(u(t), t), \ \ \phi = [1-\theta, 1+\theta]$$

where $\mathbf{x}(t) = \begin{bmatrix} e(t) \\ \dot{e}(t) \\ \psi(t) \\ \dot{\psi}(t) \end{bmatrix} \in \mathbb{R}^4$ is the linearized state.



$e(t)$ is the lateral distance to the lane center.
$\psi(t)$ is the vehicle heading relative to the path.
$u(t) \in \mathbb{R}$ is the front wheel steering angle input.

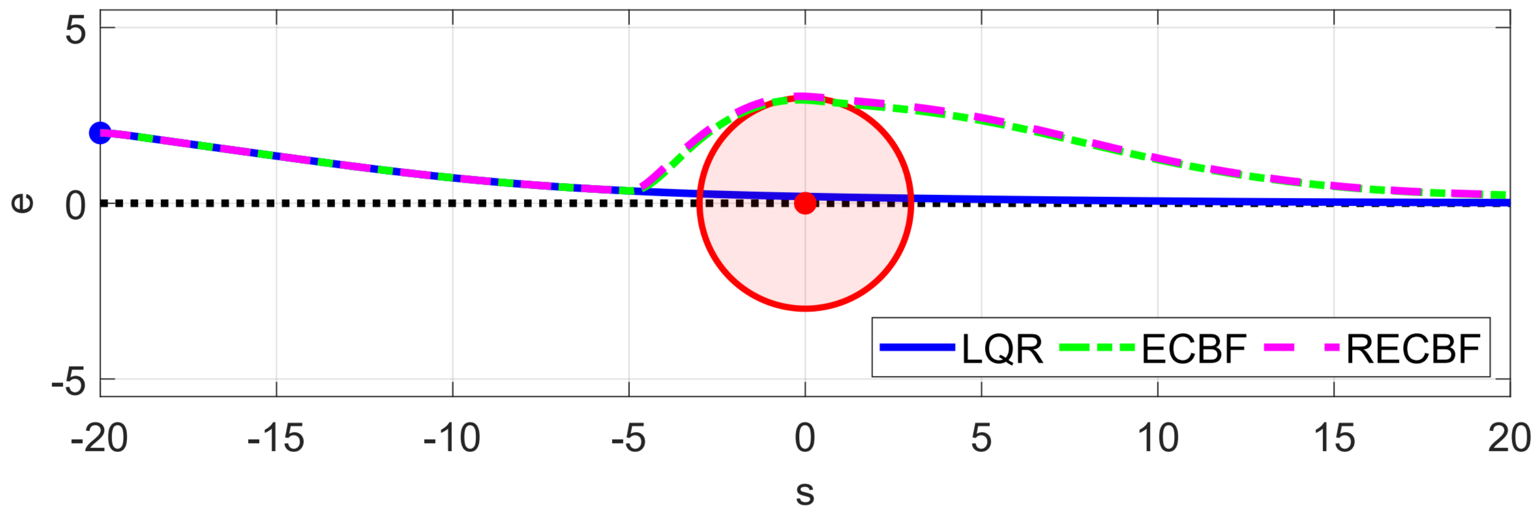
Baseline controller is reference tracking LQR with:

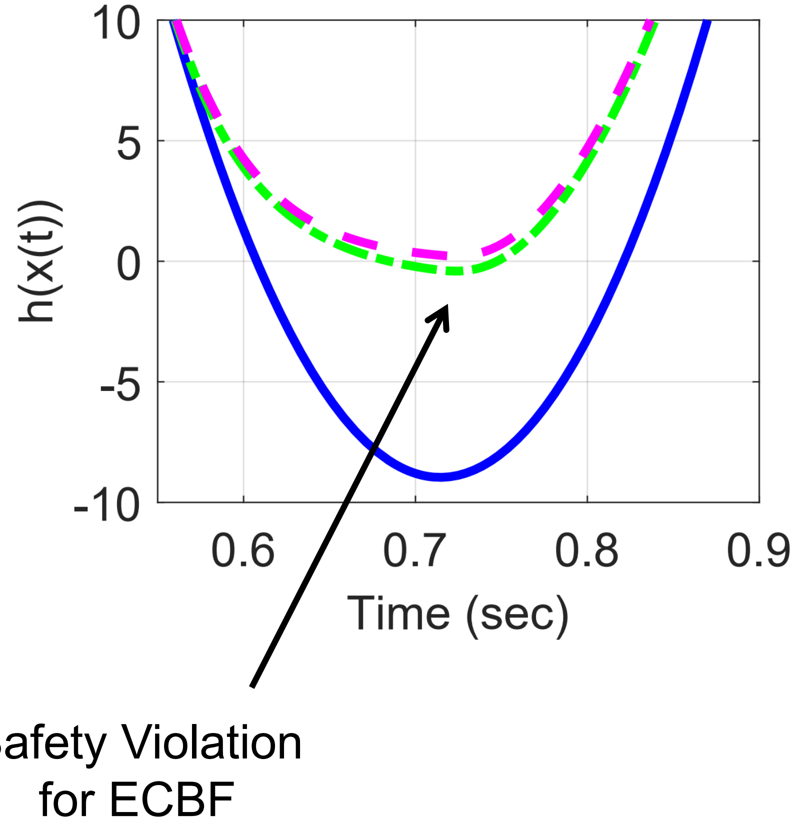
$$u_0 = \mathbf{K} \cdot (\mathbf{r} - \mathbf{x})$$

Safe-set is defined through: $\mathcal{C} \triangleq \{\mathbf{x} \in D \subset \mathbb{R}^n : h(\mathbf{x}) = e^2 + s^2 - d^2 \geq 0\}$

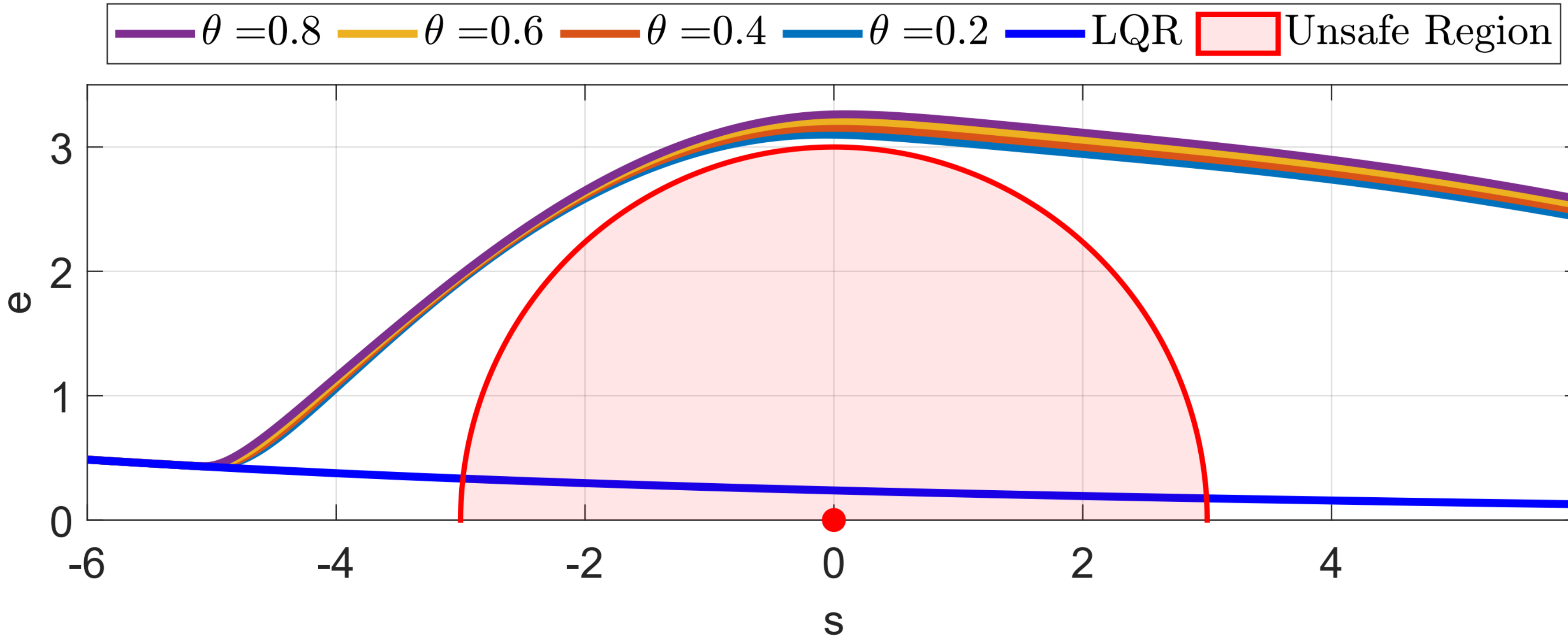
[1] Alleyne, A comparison of alternative intervention strategies for unintended roadway departure (URD) control, VSD, 1997.

Simulations with Worst-Case Uncertain Plant





Simulations with Nominal Plant



The safety-filter was designed assuming $\theta > 0$

The trajectories become more cautious around the obstacle as the uncertainty level in design increases.

Conclusion

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Thank you

Andy Packard

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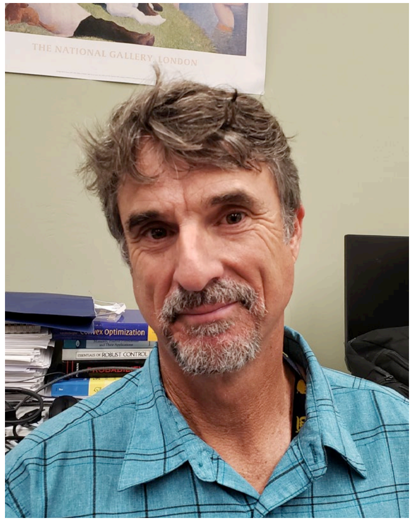

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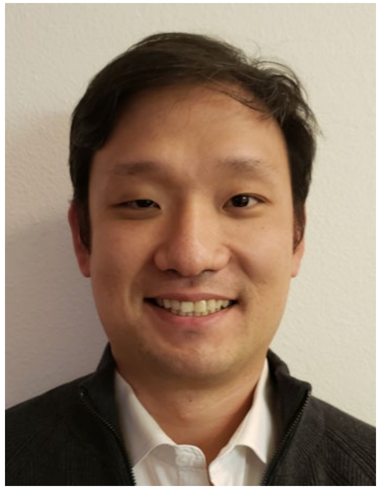

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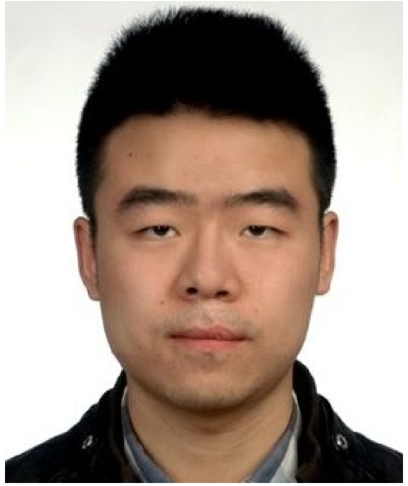

Galaxy Yin